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NUMERICAL SOLUTION OF PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNELS

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Abstract. In this paper, new numerical method based on Haar wavelet is proposed for the approximate solution of second and fourth order partial integro-differential equations with weakly singular kernels. The proposed method is based on collocation technique. The performance of the proposed method is tested by applying it to various benchmark problems from the literature. The numerical results establish the accuracy as well as efficiency of the proposed method.

Keywords: Haar wavelet, partial integro-differential equations, weakly singular kernels.
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1 Introduction

Integro-differential equations are widely used in modelling of several phenomena occurring in applied mathematics. In particular, the applications of integro-differential equations can be found in biological sciences, physics and other disciplines such as dynamics, fluid dynamics, etc. (Bloom, 1980; Holmaker, 1993; Forbes et al., 1997; Kanwal, 1971). In addition to this, integro-differential equations also arise naturally in the study of stochastic processes with jumps, or more precisely, of Levy processes.

Integro-differential equations with singularities in their kernels or solution or both also occur in several mathematical models. Integro-differential equations with weakly singular kernel are more difficult to solve and, therefore, are more challenging for numerical methods. These equations arise in heat conduction problem (Tang, 1993), elasticity and fracture mechanics (Zozulya & Gonzalez-Chi, 1999) etc.

Researchers have developed several methods for the numerical solutions of integro-differential equations with a weakly singular kernel. The examples include the spline collocation (Pedas & Tamme, 2006; Cao et al., 2007) the discrete collocation (Pedas & Tamme, 2006; Cao et al., 2007), the discrete Galerkin (Pedas & Tamme, 2008a,b), the Legendre multiwavelets method (Lakestani et al., 2011) and the piecewise polynomial collocation method with graded meshes (Parts et al., 2005). In very few papers, partial integro-differential equations with a weakly singular kernel have been considered. In Tang (1993) and Luo et al. (2015), a finite difference and a compact difference schemes are presented for partial integro-differential equations with a weakly singular kernel. In Kim & Choi, (1998), for weakly singular partial integro-differential equations a spectral collocation method is considered. Also Crank-Nicolson/quasi wavelets method (Yang et al., 2013) and quintic B-spline collocation method (Zhang et al., 2013) are used for numerical approximation of fourth order partial integro-differential equations with a weakly singular kernel and some others (Fakhar-Izadi & Dehghan, 2014; Hrusa & Renardy, 1986).

In order to determine the approximate solution by improving the accuracy of these methods of numerical solutions the calculations become more complicated and cumbersome, therefore, Haar wavelet method is a good choice to improve the accuracy of these approximate solutions. In the present paper we propose new algorithms based on the Haar wavelet which are designed for second and fourth order partial integro-differential equations with weakly singular kernels.

Wavelets, which were primarily designed for signal analysis, have also been widely used by researchers for numerical approximations in the last couple of decades. The wavelets approximation provides an efficient way to solve numerically several problems of mathematics including integration, integral equations, ordinary and partial differential equations etc.

Haar wavelet introduced by Alfred Haar in 1910 got more popularity due to its simplicity and ease of implementation. Haar wavelet expresses diverse functions in the form of combination of step functions over finite intervals. Haar wavelet is made up of pairs of piecewise constant functions and are mathematically the handiest amongst all of the wavelet functions. Among wavelets, Haar wavelet is the most popular for application to numerical approximations and many researchers have applied Haar wavelet for numerical approximations (Majak et al., 2015a,b, 2016, 2018; Aziz & Khan, 2018; Pervaiz & Aziz, 2020). In the present paper we will consider Haar wavelet approximations for the solution of second and fourth order partial integro-differential equations with weakly singular kernels.

The general form of the second order partial integro-differential equations with weakly singular kernel is given as follows:

$$w_t = \mu w_{xx} + \int_0^t (t-s)^{-p} w_{xx} ds, \quad x \in (0,1), \quad t \geq 0, \tag{1}$$

where $\mu \geq 0$ and $p \in (0,1)$. The above equation is subject to the following boundary conditions

$$w(0,t) = \alpha_1(t) \quad \text{and} \quad w(1,t) = \alpha_2(t), \quad t \geq 0, \tag{2}$$

and the initial condition

$$w(x,0) = w_0(x), \quad 0 \leq x \leq 1, \tag{3}$$

where the function $w(x,t)$ is required to be determined.

The general form of the fourth order partial integro-differential equations with weakly singular kernel is given as follows:

$$w_t + \int_0^t (t-s)^{-1/2} w_{xxxx} ds = g(x,t), \tag{4}$$

with the boundary conditions:

$$w(0,t) = 0, \quad w(1,t) = 0, \quad w_x(0,t) = 0, \quad \text{and} \quad w_x(1,t) = 0, \quad where \quad t \geq 0, \tag{5}$$

and the initial condition:

$$w(x,0) = w_0(x), \quad 0 \leq x \leq 1, \tag{6}$$

where the function $w(x,t)$ is required to be found. $g(x,t)$ is a known function.

The organization of the rest of the paper is as follows. In Section 2 Haar wavelets and their integrals are described. In Section 3 formulation of the method based on Haar wavelets is defined for second and fourth order partial integro-differential equations with weakly singular kernels. Numerical experiments are presented in Section 4 and conclusions are drawn in Section 5.

2 Haar Wavelet

The Haar wavelet family defined on the interval $[0, 1)$ is generated from a single function, called the mother wavelet, through the process of dilation and translation. The mother wavelet is a piecewise constant function having a value of 1 on the interval $[0, 0.5)$ and -1 on the interval $[0.5, 1)$. The Haar wavelet family also contains one function called the scaling function which is not generated from the mother wavelet. The scaling function is a constant function having value 1 on the whole interval $[0, 1)$. The scaling function on the interval $[0, 1)$ for Haar wavelet family is defined as follows:

$$\mathrm{h}_1(\zeta) = \begin{cases} 1 & \text{if} \quad 0 \leq \zeta \leq 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{7}$$

The remaining functions in the Haar wavelet family are defined on subintervals of $[0, 1)$ and are given as follows:

$$\mathrm{h}_i(\zeta) = \begin{cases} 1 & \text{for} \quad \zeta_1 \leq \zeta < \zeta_2 \\ -1 & \text{for} \quad \zeta_2 \leq \zeta < \zeta_3 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where

$$\zeta_1 = \frac{k}{m}, \quad \zeta_2 = \frac{k+0.5}{m}, \quad \zeta_3 = \frac{k+1}{m}, \quad i = 2, 3, \ldots, \tag{9}$$

$m = 2^j$ is an integer with $j = 0, 1, \ldots, J$, denotes the level of the wavelet and the integer k with $k = 0, 1, \ldots, m-1$ is the translation parameter. The integer J is the highest level of resolution. The relation between i, m and k is $i = 1 + m + k$.

The function $\mathrm{h}_2(\zeta)$ is named mother wavelet. The Haar wavelet functions are orthogonal to each other. We can check the orthogonality of Haar wavelet functions using the following expression,

$$\int_0^1 \mathrm{h}_k(\zeta)\mathrm{h}_l(\zeta)\mathrm{d}\zeta = 0, \quad \text{whenever} \quad k \neq l. \tag{10}$$

We introduce the following notations:

$$p_{i,1}(x) = \int_0^x h_i(\delta)\, \mathrm{d}\delta, \tag{11}$$

$$p_{i,n+1}(x) = \int_0^x p_{i,n}(\delta)\, \mathrm{d}\delta, \quad n = 1, 2, \ldots. \tag{12}$$

3 Numerical Method

We will discuss the proposed numerical method for partial integro-differential equations with weakly singular kernel using Haar wavelet collocation method. Without any loss of generality we will assume that all the equations are defined on the interval $[0, 1)$. In order to divide this interval $[0, 1)$ into subintervals the following points are used:

$$x_r = \frac{r - 0.5}{N}, \quad r = 1, 2, \ldots, N. \tag{13}$$

In Eq. (13) N is a positive integer. The points x_r, $r = 1, 2, \ldots, N$ are called collocation points. In case of Haar wavelet collocation method, the unknown function $w(x, t)$ and its derivatives in the given integro-differential equations in Eq. (1) and Eq. (4) with weakly singular kernel are estimated using Haar wavelet. A system of algebraic equations is obtained after substitution of collocation points. The solution of this system of algebraic equations gives the unknown wavelet coefficients. To calculate the approximate solution of the partial integro-differential equation with weakly singular kernel these unknown wavelet coefficients are used.

3.1 Second order partial integro differential equation with weakly singular kernel

Let the highest time derivative in Eq. (1) is approximated using Haar wavelet as given below:

$$w_t(x,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn}h_m(x)h_n(t). \tag{14}$$

By integrating Eq. (14) we have the following expressions:

$$w(x,t) = w(x,0) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn}h_m(x)p_{n,1}(t), \tag{15}$$

Let the highest space derivative in Eq. (1) is approximated using Haar wavelet as given below:

$$w_{xx}(x,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}h_m(x)h_n(t). \tag{16}$$

Integrating Eq. (16) we get the following expressions:

$$w_x(x,t) = w_x(0,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}p_{m,1}(x)h_n(t), \tag{17}$$

$$w(x,t) = w(0,t) + xw_x(0,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}p_{m,2}(x)h_n(t). \tag{18}$$

Now integrating Eq. (17) from 0 to 1 and simplifying we get

$$w_x(0,t) = w(1,t) - w(0,t) - \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}p_{m,2}(1)h_n(t). \tag{19}$$

Next after replacing this value in Eq. (18) and simplifying yields

$$w(x,t) = (1-x)w(0,t) + xw(1,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}(p_{m,2}(x) - xp_{m,2}(1))h_n(t). \tag{20}$$

By comparing Eq. (15) and Eq. (20) we get the following

$$\sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}(p_{m,2}(x)-xp_{m,2}(1))h_n(t)-\sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn}h_m(x)p_{n,1}(t) = (x-1)w(0,t)-xw(1,t)+w(x,0). \tag{21}$$

By substituting Eq. (14) and Eq. (16) in Eq. (1) we obtain the following

$$-\sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}h_m(x)\left(\mu h_n(t) + \int_0^t (t-s)^{-p}h_n(s)ds\right) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn}h_m(x)h_n(t) = 0 \tag{22}$$

Let us consider the integral

$$I = \int_0^t (t-s)^{-p}h_n(s)ds \tag{23}$$

For $n = 1$, we have

$$I = \int_0^t (t-s)^{-p}h_1(s)ds = \int_0^t (t-s)^{-p}ds = \frac{t^{1-p}}{1-p} \tag{24}$$

For $n = 2, 3, \ldots,$ we have

$$\mathrm{I} = \begin{cases} 0 & \text{for} \quad t \in [0, \zeta_1) \\ \frac{(t-\zeta_1)^{1-p}}{1-p} & \text{for} \quad t \in [\zeta_1, \zeta_2) \\ \frac{-1}{1-p}\left(2(t-\zeta_2)^{1-p} - (t-\zeta_1)^{1-p}\right) & \text{for} \quad t \in [\zeta_2, \zeta_3) \\ \frac{-1}{1-p}\left(2(t-\zeta_2)^{1-p} - (t-\zeta_1)^{1-p} - (t-\zeta_3)^{1-p}\right) & \text{for} \quad t \in [\zeta_3, 1) \end{cases} \tag{25}$$

By putting the collocation points in Eq. (21) and Eq. (22) we get a system of algebraic equations containing $2N^2$ unknown, whose solution gives the unknowns wavelet coefficients. These unknown wavelet coefficients are used to compute the approximate numerical solution of the partial integro-differential equation with weakly singular kernel.

3.2 Fourth order partial integro differential equation with weakly singular kernel

Let the highest time derivative in Eq. (4) is approximated using Haar wavelet as given below:

$$w_t(x,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn} h_m(x) h_n(t). \tag{26}$$

Integrating Eq. (26) we get the following expressions:

$$w(x,t) = w(x,0) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn} h_m(x) p_{n,1}(t), \tag{27}$$

Let the highest space derivative in Eq. (4) is approximated using Haar wavelet as given below:

$$w_{xxxx}(x,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} h_m(x) h_n(t). \tag{28}$$

Integrating Eq. (28) we get the following expressions:

$$w_{xxx}(x,t) = w_{xxx}(0,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} p_{m,1}(x) h_n(t), \tag{29}$$

$$w_{xx}(x,t) = w_{xx}(0,t) + x w_{xxx}(0,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} p_{m,2}(x) h_n(t). \tag{30}$$

$$w_x(x,t) = w_x(0,t) + x w_{xx}(0,t) + \frac{x^2}{2} w_{xxx}(0,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} p_{m,3}(x) h_n(t). \tag{31}$$

$$w(x,t) = w(0,t) + \frac{x^2}{2} w_{xx}(0,t) + \frac{x^3}{6} w_{xxx}(0,t) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} p_{m,4}(x) h_n(t). \tag{32}$$

Now integrating Eq. (30) from 0 to 1 and simplifying we get

$$-w_{xx}(0,t) - \frac{1}{2} w_{xxx}(0,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} p_{m,3}(1) h_n(t). \tag{33}$$

Now integrating Eq. (31) from 0 to 1 and simplifying we get

$$-w_{xx}(0,t) - \frac{1}{3} w_{xxx}(0,t) = 2\sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn} p_{m,4}(1) h_n(t). \tag{34}$$

Solving Eq.(33) and Eq. (34) we get

$$w_{xxx}(0,t) = -6\sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}(p_{m,3}(1) - 2p_{m,4}(1))h_n(t). \tag{35}$$

By puting Eq. (35) in Eq. (33) we get

$$w_{xx}(0,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}(3p_{m,3}(1) - 6p_{m,4}(1) - p_{m,3}(1))h_n(t). \tag{36}$$

By putting Eq. (35) and Eq. (36) in Eq. (32) we get

$$w(x,t) = \sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}\Big(x^2(p_{m,3}(1) - 3p_{m,4}(1)) - x^3(p_{m,3}(1) - 2p_{m,4}(1)) + p_{m,4}(x)\Big)h_n(t). \tag{37}$$

By comparing Eq. (27) and Eq. (37) we get the following

$$\begin{aligned}&\sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}\Big(x^2(p_{m,3}(1) - 3p_{m,4}(1)) - x^3(p_{m,3}(1) - 2p_{m,4}(1)) + p_{m,4}(x)\Big)h_n(t)\\ &-\sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn}h_m(x)p_{m,1}(t) = w(x,0).\end{aligned} \tag{38}$$

By substituting Eq. (26) and Eq. (28) in Eq. (4) we get the following

$$\sum_{m=1}^{2M}\sum_{n=1}^{2M} a_{mn}h_m(x)\left(\int_0^t (t-s)^{-1/2}h_n(s)ds\right) + \sum_{m=1}^{2M}\sum_{n=1}^{2M} b_{mn}h_m(x)h_n(t) = g(x,t) \tag{39}$$

Let us consider the integral

$$I = \int_0^t (t-s)^{-1/2}h_n(s)ds \tag{40}$$

For $n = 1$, the integral is calculated as

$$I = \int_0^t (t-s)^{-1/2}h_1(s)ds = \int_0^t (t-s)^{-1/2}ds = 2\sqrt{t} \tag{41}$$

Similarly, for $n = 2, 3, \ldots$, the integral is calculated as

$$\mathrm{I} = \begin{cases} 0 & \text{for} \quad t \in [0, \zeta_1) \\ 2\sqrt{t-\zeta_1} & \text{for} \quad t \in [\zeta_1, \zeta_2) \\ -2\left(2\sqrt{t-\zeta_2} - \sqrt{t-\zeta_1}\right) & \text{for} \quad t \in [\zeta_2, \zeta_3) \\ -2\left(2\sqrt{t-\zeta_2} - \sqrt{t-\zeta_1} - \sqrt{t-\zeta_3}\right) & \text{for} \quad t \in [\zeta_3, 1) \end{cases} \tag{42}$$

By substituting the collocation points in Eq. (38) and Eq. (39) we obtain a system of algebraic equations containing $2N^2$ unknowns, whose solution gives the unknown wavelet coefficients. These unknown wavelet coefficients are used to calculate the approximate numerical solution of the partial integro-differential equation with weakly singular kernel.

4 Numerical Experiments

To verify the efficiency and accuracy of the proposed method, some numerical experiments are presented in this section. In this section we will calculate the experimental rate of convergence $R_c(N)$ for the test problems which is defined as

$$R_c(N) = \frac{\log L_\infty(N/2) - \log L_\infty(N)}{\log 2} \tag{43}$$

Example 1. We consider the integro-differential equation:

$$w_t = \int_0^t (t-s)^{-1/2} w_{xx} ds + e^{x+t} - e^{x+t}\sqrt{\pi} erf(\sqrt{t}), \quad x \in (0,1), \quad t \geq 0 \tag{44}$$

with the boundary conditions:

$$w(0,t) = e^t \quad \text{and} \quad w(1,t) = e^{1+t}, \quad t \geq 0, \tag{45}$$

and the initial condition:

$$w(x,0) = e^x, \quad x \in [0,1]. \tag{46}$$

The analytical solution is given by

$$w(x,t) = e^{x+t}. \tag{47}$$

The maximum absolute errors L_∞, CPU time and rates of convergence for this example using different number of collocation points are given in Table 1. From the table it is clear that by increasing the number of collocation points better accuracy can be achieved. Comparison of approximate solutions with different number of collocation points with exact solution is presented in Fig. 1.

Table 1: Numerical results for example 1

N	L_∞	CPU time(s)	$R_c(N)$
2	3.0876e-002	2.172594	
4	1.3617e-002	2.584470	1.18
8	4.2124e-003	2.922334	1.69
16	1.3350e-003	6.881407	1.66

Example 2. Consider the problem

$$w_t - \int_0^t (t-s)^{-1/2} w_{xx} ds = \sin \pi x + \frac{2}{3}\pi^2 \sqrt{t}(3+2t) \sin \pi x, \quad 0 < x < 1, \quad t \geq 0 \tag{48}$$

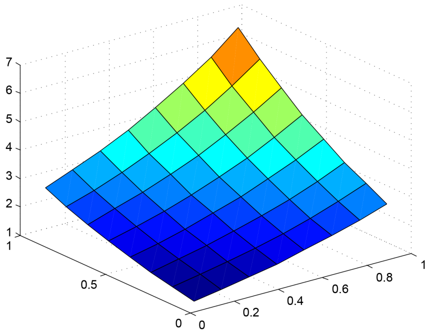
with the boundary conditions:

$$w(0,t) = 0 \quad \text{and} \quad w(1,t) = 0, \quad t \geq 0, \tag{49}$$

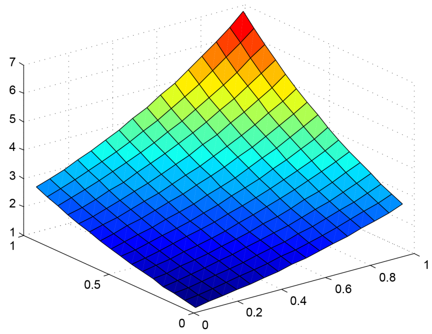
and the initial condition:

$$w(x,0) = \sin \pi x, \quad x \in [0,1]. \tag{50}$$

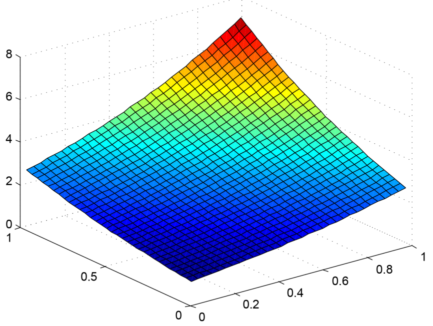
The exact solution is given by $w(x,t) = (t+1)\sin \pi x$. In Table 2 L_∞, CPU time and rate of convergence of this example for different number of collocation points are given. This table shows accuracy and efficiency of the proposed method.



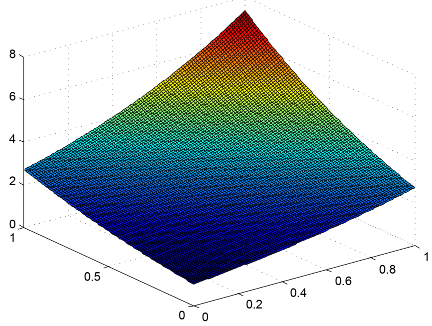
$N = 8 \times 8$



$N = 16 \times 16$



$N = 32 \times 32$



Exact solution

Figure 1: Comparison of the numerical solutions with the exact solution for Example 1

Table 2: Numerical results for Example 2

N	L_∞	CPU time(s)	$R_c(N)$
2	1.3543e-001	1.935629	....
4	5.8327e-002	2.942430	1.215
8	1.7865e-002	2.491578	1.707
16	5.2504e-003	4.785453	1.767
32	1.5654e-003	52.564796	1.746

Example 3. Consider the fourth-order problem

$$w_t + \int_0^t (t-s)^{-1/2} w_{xxxx} ds = \frac{-32}{3}\pi^4 \sqrt{t}(3+2t)\cos 2\pi x + 2\sin^2 \pi x, \quad x \in (0,1), \quad t \geq 0 \tag{51}$$

with the boundary conditions

$$w(0,t) = 0, \quad w(1,t) = 0, \quad w_x(0,t) = w_x(1,t) = 0 \quad t \geq 0, \tag{52}$$

and the initial condition

$$w(x,0) = 1 - \cos 2\pi x, \quad x \in [0,1]. \tag{53}$$

The exact solution of the problem is $w(x,t) = (2t+2)\sin^2 \pi x$. Using the proposed method, L_∞ for distinct number of collocation points for this example are presented in Table 3. From this table it is clear that better accuracy can be achieved by increasing the number of collocation points.

Table 3: Numerical results for Example 3

N	L_∞	CPU time(s)	$R_c(N)$
2	$1.7595e+001$	2.348580	....
4	$3.3899e-001$	2.122055	2.375
8	$1.0161e-001$	3.194103	1.738
16	$2.8597e-002$	8.722310	1.829
32	$1.0869e-002$	113.972244	1.396

Example 4. Consider the fourth-order problem

$$\begin{aligned} w_t + \int_0^t (t-s)^{-1/2} w_{xxxx} ds =& 2(t+1)\left(1 + 2\pi^2(1-x)x - \cos 2\pi x\right) \\ & - \frac{32}{15}\pi^4 \sqrt{t}\,(15 + 4t(5+2t)) \cos 2\pi x, \quad x \in (0,1), \quad t \geq 0 \end{aligned} \tag{54}$$

subject to the boundary conditions

$$w(0,t) = 0, \quad w(1,t) = 0, \quad w_x(0,t) = 2\pi^2(1+t)^2 \quad \text{and} \quad w_x(1,t) = -2\pi^2(t+1)^2 \quad t \geq 0, \tag{55}$$

and the initial condition

$$w(x,0) = 1 - \cos 2\pi x + 2\pi^2 x(1-x), \quad x \in [0,1]. \tag{56}$$

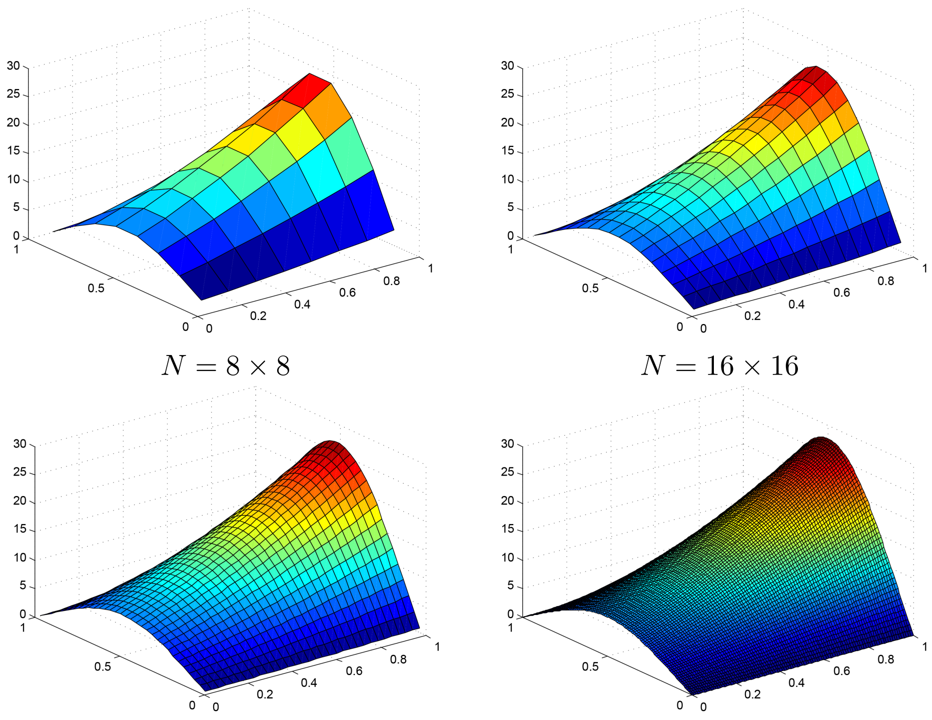
The analytical solution of the problem is given by

$$w(x,t) = (t+1)^2 \left(1 - \cos 2\pi x + 2\pi^2 x(1-x)\right). \tag{57}$$

Numerical results for this example are reported in Table 4. The good performance in terms of both accuracy and efficiency of the proposed method is evident from this table. In Fig. 2, we have shown comparison of approximate solutions with the analytical solution. It can be seen from the figure that by increasing the number of collocation points the approximate solutions are rapidly converging to the analytical solution.

Table 4: Numerical results for Example 4

N	L_∞	CPU time(s)	$R_c(N)$
2	3.0723	1.997570	....
4	6.6991e-001	1.725028	2.197
8	2.1134e-001	2.498704	1.664
16	5.6962e-002	7.479951	1.891
32	1.8671e-002	100.349032	1.609



$N = 8 \times 8$ $\quad$ $N = 16 \times 16$

$N = 32 \times 32$ $\quad$ Exact solution

Figure 2: Comparison of the numerical solutions with the exact solution for Example 4

5 Conclusion

Numerical solution of integro-differential equations of second and fourth order with a weakly singular kernel are considered in this work. For numerical approximation, Haar wavelet collocation method is applied. The proposed method is applied to four examples and good performance of the proposed method can be observed from the numerical results. It is also observed that the accuracy of the method can be increased by increasing the number of collocation points.

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